

H-DISTRIBUTIONS — AN EXTENSION OF H-MEASURES

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ABSTRACT. We prove that the $L^p(\mathbb{R}^d)$, $p \in (1, \infty)$, bound of a multiplier operator with a symbol defined on the unit sphere $S^{d-1} \subset \mathbb{R}^d$ linearly depends on the $C^\kappa(S^{d-1})$, $\kappa = [\frac{d}{2}] + 1$, bound of the symbol of multiplier operator. We use the latter properties of multiplier operators to introduce the H -distributions – an extension of the H -measures in the L^p framework. At the end of the paper, we apply the H -distributions to obtain an L^p version of the localization principle, and reprove the $L^p - L^q$ variant of the Murat-Tartar div-curl lemma.

1. INTRODUCTION

In the study of partial differential equations, quite often it is of interest to determine whether some L^p weakly convergent sequence converges strongly. Various techniques and tools have been developed for that purpose (for a review see [10]), of which we shall only mention the H -measures of Luc Tartar [31], independently introduced by Patrick Gérard [12] under the name of microlocal defect measures. H -measures proved to be very powerful tool in many fields of mathematics and physics (see e.g. [1, 2, 5, 19, 20, 14, 15, 21, 27, 28, 35] which is surely an incomplete list). The main theorem on the existence of H -measures, in an equivalent form suitable for our purposes, reads:

Theorem 1. *If scalar sequences $u_n, v_n \rightharpoonup 0$ in $L^2(\mathbb{R}^d)$, then there exist subsequences $(u_{n'}), (v_{n'})$ and a complex Radon measure μ on $\mathbb{R}^d \times S^{d-1}$ such that for every $\varphi_1, \varphi_2 \in C_0(\mathbb{R}^d)$ and every $\psi \in C(S^{d-1})$*

$$\lim_{n'} \int_{\mathbb{R}^d} \varphi_1 u_{n'} \overline{\mathcal{A}_\psi(\varphi_2 v_{n'})} dx = \langle \mu, \varphi_1 \bar{\varphi}_2 \bar{\psi} \rangle, \quad (1)$$

where \mathcal{A}_ψ is the Fourier multiplier operator with the symbol ψ :

$$\mathcal{A}_\psi u := \bar{\mathcal{F}}(\psi \hat{u}).$$

The measure μ we call the H -measure corresponding to the sequence (u_n, v_n) .

Remark 2. By applying the Plancherel theorem, the term under the limit sign in Theorem 1 takes the form

$$\int_{\mathbb{R}^d} \widehat{\varphi_1 u_{n'}} \overline{\widehat{\varphi_2 v_{n'}}} d\xi, \quad (2)$$

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where by $\hat{u}(\xi) = (\mathcal{F}u)(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} u(x) dx$ we denote the Fourier transform on \mathbb{R}^d (with the inverse $(\bar{\mathcal{F}}v)(x) := \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi} v(\xi) d\xi$). If $u_n = v_n$, μ describes the loss of strong L^2_{loc} precompactness of sequence (u_n) . Indeed, it is not difficult to see that if either (u_n) or (v_n) is strongly convergent in L^2 , then the corresponding H -measure is trivial. Conversely, for $u_n = v_n$, if the H -measure is trivial, then $u_n \rightarrow 0$ in $L^2_{loc}(\mathbb{R}^d)$ (see [6]).

In order to explain how to apply a similar idea to L^p -weakly converging sequences for $p \neq 2$, consider the integral in (1). The Cauchy-Schwartz inequality and the Plancherel theorem imply (see e.g. [31, p. 198])

$$\left| \int_{\mathbb{R}^d} \varphi_1 u_{n'}(x) \overline{\mathcal{A}_\psi(\varphi_2 v_{n'})(x)} dx \right| \leq C \|\psi\|_{C(S^{d-1})} \|\varphi_2 \varphi_1\|_{C_0(\mathbb{R}^d)},$$

where C depends on a uniform bound of $\|(u_n, v_n)\|_{L^2(\mathbb{R}^d)}$. Roughly speaking, this fact and the linearity of integral in (1) with respect to $\varphi_1 \overline{\varphi_2}$ and ψ enable us to state that the limit in (1) is a Radon measure (a functional on $C_0(\mathbb{R}^d \times S^{d-1})$). Furthermore, the bound is obtained by a simple estimate $\|\mathcal{A}_\psi\|_{L^2 \rightarrow L^2} \leq \|\psi\|_{L^\infty(\mathbb{R}^d)}$ and the fact that (u_n, v_n) is a bounded sequence in $L^2(\mathbb{R}^d; \mathbb{R}^2)$.

In [12], the question whether it is possible to extend the notion of H -measures (or microlocal defect measures in Gerard's terminology) to the L^p framework is posed. To answer the question, one necessarily needs precise bounds for the multiplier operator \mathcal{A}_ψ as the mapping from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$. The bounds are given by the famous Hörmander-Mikhlin theorem:

Theorem 3. [16, 26, 8] *Let $\phi \in L^\infty(\mathbb{R}^d)$ have partial derivatives of order less than or equal to κ , where κ is the least integer strictly greater than $d/2$ (i.e. $\kappa = \lfloor \frac{d}{2} \rfloor + 1$). If for some constant $k > 0$*

$$(\forall r > 0)(\forall \alpha \in \mathbb{N}_0^d) \quad |\alpha| \leq \kappa \implies \int_{\frac{r}{2} \leq \|\xi\| \leq r} |D_\xi^\alpha \phi(\xi)|^2 d\xi \leq k^2 r^{d-2n(\alpha)}, \quad (3)$$

then for any $p \in \langle 1, \infty \rangle$ and the associated multiplier operator T_ϕ there exists a constant C_d (depending only on the dimension d ; see [13, p. 362]) such that

$$\|T_\phi\|_{L^p \rightarrow L^p} \leq C_d p(p-1)(k + \|\phi\|_\infty). \quad (4)$$

We refer also to papers [18] and [23] and the references therein for the norm inequalities for the weighted L^p multipliers.

In Section 2, we prove that the multiplier operator T_ϕ is bounded as a mapping $L^p(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d)$ under a condition weaker than (3) involving fractional derivatives of symbol ϕ of a multiplier (see Theorem 7 and Remark 8).

By the use of estimates described above, in Section 3 we are able to introduce the H -distributions (see Theorem 11 below) — an extension of H -measures in the L^p -setting, $p > 1$. This is the main result of the paper. We conclude Section 3 by an L^p -variant of the localization principle and a proof of an $(L^p, L^{p'})$ -variant of the div-curl lemma.

For readers' convenience, some of the known theorems needed in this paper are given in the Appendix.

Remark 4. Recently, variants of H -measures with a different scaling were introduced (the parabolic H -measures [4, 5] and the ultra-parabolic H -measures [28]).

We can apply the procedure from this paper to extend the notion of such H -measures to the L^p -setting in the same fashion as for the classical H -measures given in Theorem 1.

Notation. By \mathbb{R}_+ we denote the set of non-negative real numbers; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where \mathbb{N} is the set of natural numbers; $\mathbb{Z} = \mathbb{N}_0 \cup (-\mathbb{N})$; $d \in \mathbb{N}$ denotes the dimension of the Euclidian space \mathbb{R}^d ; $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}_0^d$ and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{R}_+^d$ are multi-indices and $|\alpha| := \alpha_1 + \alpha_2 + \dots + \alpha_d$. We shall write

$$D_x^\alpha = \frac{\partial^{\alpha_1 + \alpha_2 + \dots + \alpha_d}}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}, \quad x = (x_1, \dots, x_d) \in \mathbb{R}^d, \quad \alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d.$$

Let $y \in \mathbb{R}^d$ and $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{N}_0^d$. Then

$$y^\beta := y_1^{\beta_1} y_2^{\beta_2} \dots y_d^{\beta_d}, \quad \|y\| := \left(\sum_{i=1}^d y_i^2 \right)^{1/2}.$$

A cube $J \subset \mathbb{R}^d$ is defined as $J = [a_1, b_1] \times \dots \times [a_d, b_d]$, and its dilation sJ , $s > 0$, as $sJ = [sa_1, sb_1] \times \dots \times [sa_d, sb_d]$. We denote by S^{d-1} the unit sphere in \mathbb{R}^d , while Ω stands for an open set in \mathbb{R}^d , which is not necessary bounded.

By m we denote the Lebesgue measure on \mathbb{R}^d and its subsets, while $L^p(\Omega)$ denotes the classical Lebesgue space with norm $\|\cdot\|_p$; for $p \in [1, \infty]$ the conjugated exponent will be denoted by $p' = p/(p-1)$. Furthermore, $L_{loc}^p(\Omega)$ and $L_c^p(\mathbb{R}^d)$ denote the space of locally L^p -functions (with the Fréchet topology) and the space of compactly supported L^p -functions (with the strict inductive limit topology), respectively (see [17, 34] for details). The space of tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^d)$, being the dual of $\mathcal{S}(\mathbb{R}^d)$.

By $C_c(\mathbb{R}^d)$ we denote a space of continuous compactly supported functions. $C_0(\mathbb{R}^d)$ denotes the Banach space of functions vanishing at infinity (i.e. the closure of $C_c^\infty(\mathbb{R}^d)$ in $L^\infty(\mathbb{R}^d)$), while by $C_K(\mathbb{R}^d)$ are denoted those functions in $C_c(\mathbb{R}^d)$ having their support in a given compact K . $C_K(\mathbb{R}^d)$ is a Banach space when equipped with the uniform norm.

Definition 5. Let $\phi : \mathbb{R}^d \rightarrow \mathbb{C}$ satisfy $(1 + |x|^2)^{-k/2} \phi \in L^1(\mathbb{R}^d)$ for some $k \in \mathbb{N}_0$. Then ϕ is called the Fourier multiplier on $L^p(\mathbb{R}^d)$, $p \in \langle 1, \infty \rangle$, if $\overline{\mathcal{F}}(\phi \mathcal{F}(\theta)) \in L^p(\mathbb{R}^d)$ for all $\theta \in \mathcal{S}(\mathbb{R}^d)$, and

$$\mathcal{S}(\mathbb{R}^d) \ni \theta \mapsto \overline{\mathcal{F}}(\phi \mathcal{F}(\theta)) \in L^p(\mathbb{R}^d)$$

can be extended to a continuous mapping $T_\phi : L^p(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d)$. Operator T_ϕ we call the L^p -multiplier operator with symbol ϕ .

2. HÖRMANDER-MIKHLIN THEOREM—FRACTIONAL VERSION

In order to introduce the Hörmander-Mikhlin theorem in terms of fractional derivatives, let us recall a definition of the Sobolev space of fractional order.

Definition 6. We say that $\phi \in L^2(\mathbb{R}^d)$ has fractional derivatives of order less than or equal to $\kappa \in \mathbb{R}_+$ if $\xi_1^{\alpha_1} \dots \xi_d^{\alpha_d} \mathcal{F}(\phi) \in L^2(\mathbb{R}^d)$ for every $(\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{R}_+^d$ such that $|\alpha| \leq \kappa$. Then, we write

$$D_x^\alpha \phi(x) = \overline{\mathcal{F}}(\xi_1^{\alpha_1} \xi_2^{\alpha_2} \dots \xi_d^{\alpha_d} \mathcal{F}(\phi))(x), \quad x \in \mathbb{R}^d,$$

and denote by $H^\kappa(\mathbb{R}^d)$ the corresponding vector space. We write

$$D_{x_i}^\kappa \phi(x) = \bar{\mathcal{F}}(\xi_i^\kappa \mathcal{F}(\phi))(x), \quad x \in \mathbb{R}^d, \quad \kappa \in \mathbb{N}_0,$$

and call it the i -th partial fractional derivative of order κ .

Next, we need the Littlewood-Paley diadic decomposition. Let a smooth function Θ be non-negative and satisfies $\text{supp} \Theta \subset \{\xi \in \mathbb{R}^d : 2^{-1} \leq \|\xi\| \leq 2\}$. Moreover, we assume that $\Theta(\xi) > 0$ when $2^{-\frac{1}{2}} \leq \|\xi\| \leq 2^{\frac{1}{2}}$. Let $\theta(0) := 0$ and

$$\theta(\xi) := \Theta(\xi) \left/ \sum_{j=-\infty}^{\infty} \Theta(2^{-j}\xi) \right., \quad \xi \neq 0.$$

Then, θ is non-negative, smooth, and $\text{supp} \theta \subset \{\xi \in \mathbb{R}^d : \frac{1}{2} \leq \|\xi\| \leq 2\}$, and

$$\sum_{j=-\infty}^{\infty} \theta(2^{-j}\xi) = 1, \quad \xi \neq 0.$$

Put

$$\phi_j(\xi) := \phi(\xi) \theta(2^{-j}\xi), \quad \xi \in \mathbb{R}^d, \quad j \in \mathbb{Z}. \quad (5)$$

Now, we can formulate the main theorem of this section.

Theorem 7. *Take $\phi \in L^\infty(\mathbb{R}^d)$ and define ϕ_j by (5). Suppose that there exists $\kappa > \frac{d}{2}$ such that for every $j \in \mathbb{Z}$ and every $i = 1, \dots, d$*

$$\int_{\mathbb{R}^d} |D_{\xi_i}^\kappa \phi_j(\xi)|^2 d\xi \leq p_1 2^{j(d-2\kappa)} \quad \text{and} \quad (6)$$

$$\int_{\mathbb{R}^d} |\bar{\mathcal{F}}(\phi_{j,y})(x)| dx \leq p_2 (2^j \|y\|) (2 + 2^j \|y\|)^\kappa, \quad (7)$$

where p_1, p_2 are constants independent of y .

Then ϕ is a Fourier multiplier in $L^p(\mathbb{R}^d)$ and the associated multiplier operator T_ϕ satisfies

$$\|T_\phi\|_{L^p \rightarrow L^p} \leq C, \quad (8)$$

for a constant $C > 0$.

Remark 8. Notice that in the theorem we require that only κ -th fractional derivative of ϕ satisfy (6), and that κ is an arbitrary real number greater than $[d/2]$. This means that we demand less regularity on the symbol of multiplier than in the classical Hörmander-Mikhlin theorem where it is required that $\kappa = [d/2] + 1$. Also notice that, if we assume that κ is an integer, then, if $|\alpha| = \kappa$, (6) and (7) can be obtained from the classical Hörmander-Mikhlin conditions (3) (see e.g. proof of [26, Theorem 7.5.13] or [13, Theorem 5.2.7]).

Proof: We shall pursue one of standard ideas for the proof of Theorem 3. More precisely, we shall approximate T_ϕ by a sequence of convolution operators, and then prove a uniform $L^p \rightarrow L^p$ bound for the constructed sequence.

First, notice that

$$\sum_{j=-\infty}^{+\infty} |\phi_j(\xi)| \leq 2 \|\phi\|_\infty \quad \text{a.e. } \xi \in \mathbb{R}^d,$$

since ϕ_j and ϕ_i have disjoint supports if $|i - j| \geq 2$ (see (5)).

Therefore, the multiplier operator T_{ψ_N} with the symbol

$$\psi_N(\xi) := \sum_{j=-N}^N \phi_j(\xi) \in L^2(\mathbb{R}^d), \quad \xi \in \mathbb{R}^d, \quad (9)$$

admits the following $L^2 \rightarrow L^2$ bound:

$$\|T_{\psi_N}\|_{L^2 \rightarrow L^2} \leq 2\|\phi\|_\infty. \quad (10)$$

Notice that T_{ψ_N} , $N \in \mathbb{N}$, are convolution operators with the kernels $\bar{\mathcal{F}}(\psi_N)$. Actually, the convolution operators T_{ψ_N} , $N \in \mathbb{N}$, constitute the approximating sequence announced at the beginning of the proof. In order to obtain appropriate $L^p \rightarrow L^p$, $p > 1$, bounds for the operators T_{ψ_N} , $N \in \mathbb{N}$, we need to prove that $\bar{\mathcal{F}}(\psi_N)$, $N \in \mathbb{N}$, satisfy conditions of Theorem 18. Then, we can apply the Marzinkievich-Zygmund interpolation theorem (Theorem 19 in the Appendix with $p_1 = q_1 = 1$ and $p_2 = q_2 = 2$) to obtain the bound for $\|T_{\psi_N}\|_{L^p \rightarrow L^p}$, $1 < p < 2$. Finally, using the theorem on the dual operator (cf. [36], VII.1), we obtain the bound for $\|T_{\psi_N}\|_{L^p \rightarrow L^p}$, for any $p > 1$. We provide the details in the sequel.

To realize the plan, consider cases when $2^j s > 1$ and when $2^j s \leq 1$.

Assume that $2^j s > 1$. From (6), the Cauchy-Schwartz inequality, Plancherel's theorem and the well known properties of the Fourier transform, it follows

$$\begin{aligned} \int_{\|x\| > s} |\bar{\mathcal{F}}(\phi_j)(x)| dx &\leq \left(\int_{\|x\| \geq s} \|x\|^{-2\kappa} dx \right)^{\frac{1}{2}} \left(\int_{\|x\| \geq s} \|x\|^{2\kappa} |\bar{\mathcal{F}}(\phi_j)(x)|^2 dx \right)^{\frac{1}{2}} \quad (11) \\ &\leq \left(\frac{2\pi^{d-1} s^{d-2\kappa}}{2\kappa - d} \right)^{\frac{1}{2}} \left(d^\kappa \sum_{i=1}^d \int_{\mathbb{R}^d} |x_i|^{2\kappa} |\bar{\mathcal{F}}(\phi_j)(x)|^2 dx \right)^{\frac{1}{2}} \\ &= \left(\frac{2\pi^{d-1} s^{d-2\kappa}}{2\kappa - d} \right)^{\frac{1}{2}} \left(d^\kappa \sum_{i=1}^d \int_{\mathbb{R}^d} |D_{\xi_i}^\kappa \phi_j(\xi)|^2 d\xi \right)^{\frac{1}{2}} \\ &\leq p_3 s^{\frac{d}{2} - \kappa} 2^{j(\frac{d}{2} - \kappa)}, \end{aligned}$$

where $p_3 > 0$ does not depend on j . Since we assumed $2^j s > 1$, it follows $(2^j s)^{d/2 - \kappa} < 1$ (recall that $d/2 - \kappa < 0$) and estimate (11) is sufficient to control ψ_N .

If $2^j s < 1$ we need estimate (7). Indeed, assume that $2^j s < 1$ and $\|y\| \leq \frac{s}{2}$. It follows from (7) that

$$\int_{\mathbb{R}^d} |\bar{\mathcal{F}}(\phi_j)(x - y) - \bar{\mathcal{F}}(\phi_j)(x)| dx \leq \frac{p_2}{2} 3^\kappa (2^j s), \quad y \in \mathbb{R}^d. \quad (12)$$

From here and (11), it follows that for every $y \in \mathbb{R}^d$:

$$\int_{\|x\| \geq s} |\bar{\mathcal{F}}(\psi_N)(x - y) - \bar{\mathcal{F}}(\psi_N)(x)| dx \leq p_5 \sum_{j=-\infty}^{\infty} \min\{(2^j s)^{\frac{1}{2}d - \kappa}, 2^j s\}, \quad (13)$$

where p_5 is independent on s . Furthermore, since $\sum_{j=-\infty}^{\infty} \min\{(2^j s)^{\frac{1}{2}d - \kappa}, 2^j s\}$ is bounded in s , (13) implies that:

$$\int_{\|x\| \geq s} |\bar{\mathcal{F}}(\psi_N)(x - y) - \bar{\mathcal{F}}(\psi_N)(x)| dx \leq p_6, \quad \|y\| \leq s/2, \quad (14)$$

where p_6 is independent on s or k . Introducing here the change of variables $x = tu$ and taking ty and ts in the place of y and s , respectively, we immediately obtain

$$\int_{\|x\| \geq s} |U_t(\bar{\mathcal{F}}(\psi_N))(x - y) - U_t(\bar{\mathcal{F}}(\psi_N))(x)| dx \leq p_6, \quad \|y\| \leq s/2, \quad (15)$$

where $U_t(\psi_N)$ is given in Definition 17, i.e. $\bar{\mathcal{F}}(\psi_N)$ is a singular kernel of exponent 1. From (10) and (15) we see that conditions of Theorem 18 are fulfilled for the convolution operator with the kernel $\bar{\mathcal{F}}(\psi_N)$, and conclude that there exists a constant p_7 such that for every $a > 0$ and every $f \in L^1(\mathbb{R}^d)$,

$$m(\{x \in \mathbb{R}^d : |\bar{\mathcal{F}}(\psi_N) \star f(x)| > a\}) \leq p_7 a^{-1} \|f\|_{L^1(\mathbb{R}^d)}. \quad (16)$$

Next, it follows from (10) and Theorem 20 that for every $f \in L^2(\mathbb{R}^d)$ and every $a > 0$,

$$m(\{x : |\bar{\mathcal{F}}(\psi_N) \star f(x)| > a\})^{1/2} \leq 2\|\phi\|_{\infty} a^{-1} \|f\|_{L^2(\mathbb{R}^d)}. \quad (17)$$

Finally, combining (16) and (17) with Theorem 19, we conclude that there exists $p_8 > 0$ such that

$$\|\bar{\mathcal{F}}(\psi_N) \star f(x)\|_{L^p(\mathbb{R}^d)} \leq p_8 \|f\|_{L^p(\mathbb{R}^d)}, \quad f \in L^p(\mathbb{R}^d), \quad (18)$$

where p_8 depends on $p \in (0, 1)$.

Next, notice that the dual operator (cf. [36], VII.1) T'_φ of a bounded convolution operator $T_\varphi : L^p(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d)$ ($p \in \langle 1, \infty \rangle$) with the kernel φ is given by $T'_\varphi = T_{\tilde{\varphi}}$, $\tilde{\varphi}(\xi) = \varphi(-\xi)$, and that

$$\|T_\varphi\|_{L^p \rightarrow L^p} = \|T'_\varphi\|_{L^{p'} \rightarrow L^{p'}} = \|T_\varphi\|_{L^{p'} \rightarrow L^{p'}}.$$

From here, we see that (18) holds for any $p > 2$.

Next, since

$$\|T_{\psi_N}(f) - T_\phi(f)\|_{L^2} \rightarrow 0 \quad \text{as } N \rightarrow \infty,$$

we know that there exists a subsequence $(T_{\psi_{N_j}})$, $j \in \mathbb{N}$, such that $T_{\psi_{N_j}} \rightarrow T_\phi$ a.e. in \mathbb{R}^d as $j \rightarrow \infty$. Therefore, by Fatou's lemma and (18), it follows

$$\|T_\phi(f)\|_{L^p(\mathbb{R}^d)} \leq \liminf_{j \rightarrow \infty} \|T_{\psi_{N_j}}\|_p \leq p_8 \|f\|_{L^p(\mathbb{R}^d)}, \quad f \in L^p(\mathbb{R}^d),$$

for any $p > 1$. □

3. A GENERALIZATION OF H -MEASURES

We have already seen (Remark 2) that an H -measure μ corresponding to a sequence (u_n) in $L^2(\mathbb{R}^d)$ can describe its loss of strong compactness [12, 31]. We would like to introduce a similar notion describing the loss (at least in L^1_{loc}) of strong compactness for a sequence weakly converging in $L^p(\mathbb{R}^d)$ (in this section we consider only $p \in \langle 1, \infty \rangle$). Our extension is motivated by the following lemma and its corollary.

Lemma 9. [9, Lemma 7] *For $l \in \mathbb{R}^+$ and $u \in \mathbb{R}$ denote*

$$T_l(u) = \begin{cases} 0, & u > l \\ u, & u \in [-l, l] \\ 0, & u < -l. \end{cases} \quad (19)$$

Assume that a sequence (u_n) of measurable functions on $\Omega \subset \mathbb{R}^d$ is such that for some $s > 0$

$$\sup_{n \in \mathbb{N}} \int_{\Omega} |u_n|^s dx < \infty.$$

Suppose further that for each fixed $l > 0$ the sequence of truncated functions

$$(T_l(u_n))_n \text{ is precompact in } L^1(\Omega).$$

Then, there exists a measurable function u such that on a subsequence

$$u_{n_k} \rightarrow u \text{ in measure.}$$

Corollary 10. *The subsequence in Lemma 9 satisfies*

$$u_{n_k} \rightarrow u \text{ strongly in } L^1_{loc}(\Omega).$$

Proof: By the Fatou lemma (the form requiring only convergence in measure [11]), we conclude that $u \in L^p(\Omega)$. Furthermore, for a compact $K \subseteq \Omega$ on the limit $k \rightarrow \infty$ we have

$$\begin{aligned} \int_K |u_{n_k} - u| dx &= \int_{\{|u_{n_k} - u| > 1/k\} \cap K} |u_{n_k} - u| dx + \int_{\{|u_{n_k} - u| \leq 1/k\} \cap K} |u_{n_k} - u| dx \\ &\leq (m(\{|u_{n_k} - u| > 1/k\} \cap K))^{p'} \int_K |u_{n_k} - u|^p dx + m(K)/k \rightarrow 0. \end{aligned}$$

□

So, we see that if we want to analyze the strong L^1_{loc} compactness for a sequence (u_n) weakly converging to zero in $L^p(\mathbb{R}^d)$, it is enough to inspect how the truncated sequences $(v_{n,l})_n := (T_l(u_n))_n$ behave. Furthermore, notice that it is not enough to consider $(v_{n,l})_{n,l}$ separately since this would force us to estimate $u_n - v_{n,l}$ which is usually not easy. For instance, consider a sequence (u_n) weakly converging to zero in $L^p(\mathbb{R}^d)$, and solving the following family of problems:

$$\sum_{i=1}^d \partial_{x_i} (A_i(x) u_n(x)) = f_n(x), \quad (20)$$

where $A_i \in C_0(\mathbb{R}^d)$ and $f_n \rightarrow 0$ strongly in the Sobolev space $H^{-1}(\mathbb{R}^d)$. When dealing with the latter equation it is standard to multiply (20) by $\mathcal{A}_{\frac{\psi}{|\xi|}}(\phi u_n)$ (in the duality product of $H^{-1}(\mathbb{R}^d)$ by $H^1(\mathbb{R}^d)$), for $\phi \in C_0(\mathbb{R}^d)$, where $\mathcal{A}_{\frac{\psi}{|\xi|}}$ is the multiplier operator with symbol $\frac{\psi(\xi/|\xi|)}{|\xi|}$, $\psi \in C(S^{d-1})$, and then pass to limit $n \rightarrow \infty$ (see e.g. [3, 29]). If $u_n \in L^2(\mathbb{R}^d)$, we can apply standard H -measures to describe the defect of compactness for (u_n) .

If we instead take $u_n \in L^p(\mathbb{R}^d)$, for $p < 2$, we can try to rewrite (20) in the form

$$\sum_{i=1}^d \partial_{x_i} (A_i(x) T_l(u_n)(x)) = f_n(x) + \sum_{i=1}^d \partial_{x_i} (A_i(x) (T_l(u_n)(x) - u_n(x))),$$

and, similarly as before, to multiply (20) by $\mathcal{A}_{\frac{\psi}{|\xi|}}(\phi T_l(u_n))$. Unfortunately, we are not able to control the right-hand side of such an expression and we need to change the strategy. In view of the latter considerations, we formulate the following theorem.

Theorem 11. *If $u_n \rightharpoonup 0$ in $L^p(\mathbb{R}^d)$ and $v_n \xrightarrow{*} v$ in $L^\infty(\mathbb{R}^d)$, then there exist subsequences $(u_{n'})$, $(v_{n'})$ and a complex valued distribution $\mu \in \mathcal{D}'(\mathbb{R}^d \times S^{d-1})$ of order not more than $\kappa = [d/2] + 1$, such that for every $\varphi_1, \varphi_2 \in C_0(\mathbb{R}^d)$ and $\psi \in C^\kappa(S^{d-1})$ we have:*

$$\begin{aligned} \lim_{n' \rightarrow \infty} \int_{\mathbb{R}^d} \overline{\mathcal{A}_\psi(\varphi_1 u_{n'})(x)} (\varphi_2 v_{n'})(x) dx &= \lim_{n' \rightarrow \infty} \int_{\mathbb{R}^d} (\varphi_1 u_{n'})(x) \mathcal{A}_\psi(\varphi_2 v_{n'})(x) dx \\ &= \langle \mu, \varphi_1 \varphi_2 \psi \rangle, \end{aligned} \quad (21)$$

where $\mathcal{A}_\psi : L^p(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d)$ is a multiplier operator with symbol $\psi \in C^\kappa(S^{d-1})$.

We call the functional μ the *H-distribution* corresponding to (a subsequence of) (u_n) and (v_n) .

Remark 12. Notice that, unlike to what was the case with *H*-measures, it is not possible to write (21) in a form similar to (2) since, according to the Hausdorff-Young inequality, $\|\mathcal{F}(u)\|_{L^{p'}(\mathbb{R}^d)} \leq C\|u\|_{L^p(\mathbb{R}^d)}$ only if $1 < p < 2$. This means that we are not able to estimate $\|\mathcal{F}(\varphi_2 v_n)\|_{L^q(\mathbb{R}^d)}$, $q > 2$, which would appear from (21) when rewriting it in a form similar to (2).

In order to prove the theorem, we need a consequence of Tartar's First commutation lemma [31, Lemma 1.7]. First, for $a \in C^\kappa(S^{d-1})$ and $b \in C_0(\mathbb{R}^d)$ define the Fourier multiplier operator \mathcal{A}_ψ and the operator of multiplication B on $L^p(\mathbb{R}^d)$, by the formulae:

$$\mathcal{F}(\mathcal{A}_\psi u)(\xi) = a\left(\frac{\xi}{|\xi|}\right) \mathcal{F}(u)(\xi), \quad (22)$$

$$Bu(x) = b(x)u(x). \quad (23)$$

Notice that a satisfies the conditions of the Hörmander-Mikhlin theorem (see [30, Sect. 3.2, Example 2]). Therefore, \mathcal{A}_ψ and B are bounded operators on $L^p(\mathbb{R}^d)$, for any $p \in \langle 1, \infty \rangle$. We are interested in the properties of their commutator, $C = \mathcal{A}_\psi B - B\mathcal{A}_\psi$.

Lemma 13. *Let (v_n) be bounded in both $L^2(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$, and such that $v_n \rightharpoonup 0$ in the sense of distributions. Then the sequence (Cv_n) strongly converges to zero in $L^q(\mathbb{R}^d)$, for any $q \in \langle 2, \infty \rangle$.*

Proof: First, notice that according to the classical interpolation inequality:

$$\|Cv_n\|_q \leq \|Cv_n\|_2^\alpha \|Cv_n\|_p^{1-\alpha}, \quad (24)$$

for any $\alpha \in \langle 0, 1 \rangle$ and $1/q = \alpha/2 + (1 - \alpha)/p$. As C is a compact operator on $L^2(\mathbb{R}^d)$ by the First commutation lemma, while C is bounded on $L^p(\mathbb{R}^d)$, from (24) we get the claim. \square

Proof of Theorem 11: Initially, we prove the first inequality in (21). For that purpose, we approximate the sequence $(u_{n'})$ by a sequence of L^2 functions $(u_{n'}^\varepsilon)$ such that

$$\lim_{\varepsilon \rightarrow \infty} \|u_{n'}^\varepsilon - u_{n'}\|_{L^p} = 0 \quad (25)$$

uniformly with respect to $n' \in \mathbb{N}$. It holds from (21), Plancherel's theorem and Hörmander-Mikhlin's theorem:

$$\begin{aligned}
& \lim_{n' \rightarrow \infty} \int_{\mathbb{R}^d} \overline{\mathcal{A}_\psi(\varphi_1 u_{n'})(x)} (\varphi_2 v_{n'})(x) dx \\
&= \lim_{n' \rightarrow \infty} \int_{\mathbb{R}^d} \overline{\mathcal{A}_\psi(\varphi_1(u_{n'} - u_{n'}^\varepsilon))(x)} (\varphi_2 v_{n'})(x) dx + \lim_{n' \rightarrow \infty} \int_{\mathbb{R}^d} \overline{\mathcal{A}_\psi(\varphi_1 u_{n'}^\varepsilon)(x)} (\varphi_2 v_{n'})(x) dx \\
&= \lim_{n' \rightarrow \infty} \mathcal{O}(\|u_{n'} - u_{n'}^\varepsilon\|_{L^p}) + \lim_{n' \rightarrow \infty} \int_{\mathbb{R}^d} \overline{\mathcal{F}(\varphi_1 u_{n'}^\varepsilon)(\xi)} \psi(\xi/|\xi|) \mathcal{F}(\varphi_2 v_{n'})(\xi) d\xi \\
&= \lim_{n' \rightarrow \infty} \mathcal{O}(\|u_{n'} - u_{n'}^\varepsilon\|_{L^p}) + \lim_{n' \rightarrow \infty} \int_{\mathbb{R}^d} (\varphi_1 u_{n'}^\varepsilon)(x) \mathcal{A}_\psi(\varphi_2 v_{n'})(x) dx \\
&= \lim_{n' \rightarrow \infty} \int_{\mathbb{R}^d} (\varphi_1 u_{n'})(x) \mathcal{A}_\psi(\varphi_2 v_{n'})(x) dx,
\end{aligned}$$

according to (25). We now pass to the second inequality in (21).

Since $u_n \rightharpoonup 0$ in $L^p(\mathbb{R}^d)$, while for $v \in L^\infty(\mathbb{R}^d)$ we have $\varphi_1 \mathcal{A}_\psi(\varphi_2 v) \in L^{p'}(\mathbb{R}^d)$, according to the Hörmander-Mikhlin theorem for any $\varphi_1, \varphi_2 \in C_c(\mathbb{R}^d)$ and $\psi \in C^\kappa(S^{d-1})$ it follows that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \varphi_1 u_n \mathcal{A}_\psi(\varphi_2 v) dx = 0.$$

We can write $\mathbb{R}^d = \bigcup_{l \in \mathbb{N}} K_l$, where K_l form an increasing family of compact sets (e.g. closed balls around the origin of radius l); therefore $\text{supp } \varphi_2 \subseteq K_l$ for some $l \in \mathbb{N}$. We have:

$$\begin{aligned}
\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \varphi_1 u_n \mathcal{A}_\psi(\varphi_2 v_n) dx &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \varphi_1 u_n \mathcal{A}_\psi[\varphi_2 \chi_l(v_n - v)] dx \\
&= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \varphi_1 \varphi_2 u_n \mathcal{A}_\psi(\chi_l(v_n - v)) dx \\
&= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \varphi_1 \varphi_2 u_n \mathcal{A}_\psi(\chi_l v_n) dx,
\end{aligned}$$

where χ_l is the characteristic function of K_l . In the second equality we have used Lemma 13.

This allows us to express the above integrals as bilinear functionals, after denoting $\varphi = \varphi_1 \varphi_2$:

$$\mu_{n,l}(\varphi, \psi) = \int_{\mathbb{R}^d} \varphi u_n \mathcal{A}_\psi(\chi_l v_n) dx. \quad (26)$$

Furthermore, $\mu_{n,l}$ is bounded by $\tilde{C} \|\varphi\|_{C_0(\mathbb{R}^d)} \|\psi\|_{C^\kappa(S^{d-1})}$, as according to the Hölder inequality and Theorem 3:

$$|\mu_{n,l}(\varphi, \psi)| \leq \|\varphi u_n\|_p \|\mathcal{A}_\psi(\chi_l v_n)\|_{p'} \leq \tilde{C} \|\psi\|_{C^\kappa(S^{d-1})} \|\varphi\|_{C_0(\mathbb{R}^d)},$$

where the constant \tilde{C} depends on $L^p(K_l)$ -norm and $L^{p'}(K_l)$ -norm of the sequences (u_n) and (v_n) , respectively.

For each $l \in \mathbb{N}$ we can apply Lemma 22 below to obtain operators $B^l \in \mathcal{L}(C_{K_l}(\mathbb{R}^d); (C^\kappa(S^{d-1}))')$. Furthermore, we can for the construction of B^l start with a defining subsequence for B^{l-1} , so that the convergence will remain valid on $C_{K_{l-1}}(\mathbb{R}^d)$, in such a way obtaining that B^l is an extension of B^{l-1} .

This allows us to define the operator B on $C_c(\mathbb{R}^d)$: for $\varphi \in C_c(\mathbb{R}^d)$ we take $l \in \mathbb{N}$ such that $\text{supp } \varphi \subseteq K_l$, and set $B\varphi := B^l\varphi$. Because of the above mentioned

extension property, this definition is good, and we have a bounded operator:

$$\|B\varphi\|_{(C^\kappa(S^{d-1}))'} \leq \tilde{C}\|\varphi\|_{C_0(\mathbb{R}^d)} .$$

In such a way we got a bounded linear operator B on the space $C_c(\mathbb{R}^d)$ equipped with the uniform norm; the operator can be extended to its completion, the Banach space $C_0(\mathbb{R}^d)$.

Now we can define $\mu(\varphi, \psi) := \langle B\varphi, \psi \rangle$, which satisfies (21).

We can restrict B to an operator \tilde{B} defined only on $C_c^\infty(\mathbb{R}^d)$; as the topology on $C_c^\infty(\mathbb{R}^d)$ is stronger than the one inherited from $C_0(\mathbb{R}^d)$, the restriction remains continuous. Furthermore, $(C^\kappa(S^{d-1}))'$ is the space of distributions of order κ , which is a subspace of $\mathcal{D}'(S^{d-1})$. In such a way we have a continuous operator from $C_c^\infty(\mathbb{R}^d)$ to $\mathcal{D}'(S^{d-1})$, which by the Schwartz kernel theorem can be identified to a distribution from $\mathcal{D}'(\mathbb{R}^d \times S^{d-1})$ (for details cf. [17, Ch. VI]). \square

Remark that Theorem 11 also holds if we assume that (v_n) is a bounded sequence in $L^{p'}(\mathbb{R}^d)$, for $p' > 1$ such that $1/p + 1/p' = 1$ (we shall need this fact in Theorem 16. Still, usual problem in applications is to prove that a weakly convergent sequence is, at the same time, strongly convergent (see e.g. [1, 7, 27, 29]). In view of Corollary 10, in order to prove strong L_{loc}^1 strong convergence of a weakly convergent sequence, the given version of Theorem 11 is sufficient. Indeed, assume that $u_n \rightharpoonup 0$ in $L^p(\mathbb{R}^d)$. Denote $v_n^l = T_l(u_n)$ and assume that we are able to prove that the H -distribution μ^l corresponding to subsequences $(u_{n'})$ and $(v_{n'}^l)$ is identically equal to zero for each $l \in \mathbb{N}$. In that case, taking $\psi = 1$, $\varphi_1 = \varphi_2 = \varphi$ in (21), we have:

$$\begin{aligned} \lim_{n' \rightarrow \infty} \int_{\mathbb{R}^d} \varphi u_{n'} \mathcal{A}_1(\varphi v_{n'}^l) dx &= \lim_{n' \rightarrow \infty} \int_{\mathbb{R}^d} \varphi^2 u_{n'} T_l(u_{n'}) dx \\ &= \lim_{n' \rightarrow \infty} \int_{\mathbb{R}^d} \varphi^2 |T_l(u_{n'})|^2 dx = 0 . \end{aligned}$$

This implies that for any fixed $l \in \mathbb{N}$ we have $v_{n'}^l \rightarrow 0$ strongly L_{loc}^2 , implying the same convergence in L_{loc}^1 . Now by Corollary 10 we conclude that $u_n \rightarrow 0$ in L_{loc}^1 . Comparing the latter with Remark 2, we see that H -distributions are a proper generalization of H -measures. Actually, the following localization principle holds (see also [31, Theorem 1.6] and [3, Theorem 2]).

Theorem 14. *Consider (20), under the assumptions that $u_n \rightharpoonup 0$ in $L^p(\mathbb{R}^d)$, for some $p > 1$, and $f_n \rightarrow 0$ in $W^{-1,q}(\mathbb{R}^d)$, for some $q \in \langle 1, d \rangle$. Take an arbitrary sequence (v_n) bounded in $L^\infty(\mathbb{R}^d)$, and by μ denote the H -distribution corresponding to some subsequences of sequences (u_n) and (v_n) . Then*

$$\sum_{i=1}^d A_i(x) \xi_i \mu(x, \xi) = 0 , \quad (27)$$

in the sense of distributions on $\mathbb{R}^d \times S^{d-1}$, the function $(x, \xi) \mapsto \sum_{i=1}^d A_i(x) \xi_i$ being the symbol of the linear partial differential operator with C_0 coefficients.

Proof: In order to prove the theorem, we need a particular multiplier, the so called (Marcel) Riesz potential $I_1 := \mathcal{A}_{|2\pi\xi|^{-1}}$, and the Riesz transforms $R_j := \mathcal{A}_{\frac{\xi_j}{|\xi|}}$ [30,

V.1,2]. We note that [id., V.2.3]

$$\int I_1(\phi) \partial_j g = \int (R_j \phi) g, \quad g \in \mathcal{S}(\mathbb{R}^d). \quad (28)$$

From here, using a density argument and the fact that R_j is bounded from $L^p(\mathbb{R}^d)$ to itself, we conclude that $\partial_j I_1(\phi) = -R_j(\phi)$, for $\phi \in L^p(\mathbb{R}^d)$.

We should prove that the H-distribution corresponding to (subsequences of) (u_n) and (v_n) satisfies (27). To this end, take the following sequence of test functions:

$$\phi_n := \varphi_1(I_1 \circ \mathcal{A}_{\psi(\xi/|\xi|)})(\varphi_2 v_n),$$

where $\varphi_1, \varphi_2 \in C_c^\infty(\mathbb{R}^d)$ and $\psi \in C^\kappa(S^{d-1})$, $\kappa = [d/2] + 1$. Then, apply the right-hand side of (20), which converges strongly to 0 in $W^{-1,q}(\mathbb{R}^d)$ by the assumption, to a weakly converging sequence (ϕ_n) in the dual space $W^{1,q'}(\mathbb{R}^d)$.

We can do that since (ϕ_n) is a bounded sequence in $W^{1,r}(\mathbb{R}^d)$ for any $r \in \langle 1, \infty \rangle$.

Indeed, $\mathcal{A}_{\psi}(\varphi_2 v_n)$ is bounded in any $L^r(\mathbb{R}^d)$ ($r > 1$). By the well known fact [30, Theorem V.1] that I_1 is bounded from $L^q(\mathbb{R}^d)$ to $L^{q^*}(\mathbb{R}^d)$, for $q \in \langle 1, d \rangle$ and $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{d}$, ϕ_n is bounded in $L^{q^*}(\mathbb{R}^d)$, for all sufficiently large q^* . Then, take $q^* \geq r$ and due to the compact support of φ_1 we have that L^{q^*} boundedness implies the same in L^r . On the other hand, R_j is bounded from $L^r(\mathbb{R}^d)$ to itself, for any $r \in \langle 1, \infty \rangle$, thus $\partial_j(\varphi_1(I_1 \circ \mathcal{A}_{\psi(\xi/|\xi|)})(\varphi_2 v_n))$ is bounded in $L^r(\mathbb{R}^d)$.

Therefore we have (the sequence is bounded and 0 is the only accumulation point, so the whole sequence converges to 0)

$$\lim_{n \rightarrow \infty} W^{-1,q}(\mathbb{R}^d) \langle f_n, \phi_n \rangle_{W^{1,q'}(\mathbb{R}^d)} = 0. \quad (29)$$

Concerning the left-hand side of (20), according to (28) one has

$$\begin{aligned} & W^{-1,q}(\mathbb{R}^d) \langle \sum_{j=1}^d \partial_j (A_j u_n), \phi_n \rangle_{W^{1,q'}(\mathbb{R}^d)} \\ &= \int_{\mathbb{R}^d} \sum_{j=1}^d \varphi_1 A_j u_n \mathcal{A}_{\frac{\xi_j}{|\xi|} \psi(\xi/|\xi|)}(\varphi_2 u_n) dx - \int_{\mathbb{R}^d} \partial_j \varphi_1 \sum_{j=1}^d A_j u_n (I_1 \circ \mathcal{A}_{\psi(\xi/|\xi|)})(\varphi_2 v_n) dx. \end{aligned} \quad (30)$$

The first term on the right is of the form of the right-hand side of (21). The integrand in the second term is supported in a fixed compact and weakly converging to 0 in L^p , so strongly in $W^{-1,r'}$, where r is such that $p = r^*$ (i.e. $r = dp/(d-p)$). Of course, the argument giving the boundedness of ϕ_n in $W^{1,q'}(\mathbb{R}^d)$ above applies also to r instead of q' .

Therefore, from (29) and (30) we conclude (27). \square

Remark 15. Notice that the assumption of the strong convergence of f_n in $W^{-1,q}(\mathbb{R}^d)$ can be relaxed to local convergence, as in the proof we used a cutoff function φ_1 .

We conclude the paper by another corollary of Theorem 11 - the well known Murat-Tartar div-curl lemma in the (L^p, L^q) -setting [24, 25, 32].

Theorem 16. *Let (u_n^1, u_n^2) and (v_n^1, v_n^2) be vector valued sequences converging to zero weakly in $L^p(\mathbb{R}^2)$ and $L^{p'}(\mathbb{R}^2)$, respectively.*

Assume the sequence $(\partial_x u_n^1 + \partial_y u_n^2)$ is bounded in $L^p(\mathbb{R}^2)$, and the sequence $(\partial_y v_n^1 - \partial_x v_n^2)$ is bounded in $L^{p'}(\mathbb{R}^2)$, $(x, y) \in \mathbb{R}^2$.

Then, the sequence $(u_n v_n)$ converges to zero in the sense of distributions (vaguely).

Proof: Denote by μ^{ij} the H -distribution corresponding to the sequences (u_n^i) and (v_n^j) , $i, j = 1, 2$ (see the comment after proof of Theorem 11).

Since $(\partial_x u_n^1 + \partial_y u_n^2)$ is bounded in $L^p(\mathbb{R}^2)$, and $(\partial_y v_n^1 - \partial_x v_n^2)$ is bounded in $L^{p'}(\mathbb{R}^2)$, it is not difficult to see that $\partial_x u_n^1 + \partial_y u_n^2 \rightharpoonup 0$ in L^p and $\partial_y v_n^1 - \partial_x v_n^2 \rightharpoonup 0$ in $L^{p'}$. Now, from the compactness properties of the Riesz potential I_1 (see proof of the previous theorem), we conclude that for every $\varphi \in C_c(\mathbb{R}^2)$ the following limits hold in $L^p(\mathbb{R}^2)$ and $L^{p'}(\mathbb{R}^2)$, respectively:

$$\mathcal{A}_{\psi(\xi/|\xi|)\frac{\xi_1}{|\xi|}}(\varphi u_n^1) + \mathcal{A}_{\psi(\xi/|\xi|)\frac{\xi_2}{|\xi|}}(\varphi u_n^2) = \mathcal{A}_{\frac{\psi(\xi/|\xi|)}{|\xi|}}(\partial_x(\varphi u_n^1) + \partial_y(\varphi u_n^2)) \rightarrow 0, \quad (31)$$

$$\mathcal{A}_{\psi(\xi/|\xi|)\frac{\xi_2}{|\xi|}}(\varphi v_n^1) - \mathcal{A}_{\psi(\xi/|\xi|)\frac{\xi_1}{|\xi|}}(\varphi v_n^2) = \mathcal{A}_{\frac{\psi(\xi/|\xi|)}{|\xi|}}(\partial_y(\varphi v_n^1) - \partial_x(\varphi v_n^2)) \rightarrow 0. \quad (32)$$

Multiplying (31) first by φv_n^1 and then by φv_n^2 , integrating over \mathbb{R}^2 and letting $n \rightarrow \infty$, we conclude from Theorem 11 due to arbitrariness of ψ and φ :

$$\xi_1 \mu^{11} + \xi_2 \mu^{21} = 0, \quad \xi_1 \mu^{12} + \xi_2 \mu^{22} = 0, \quad (33)$$

Similarly, from (32), we obtain:

$$\xi_2 \mu^{11} - \xi_1 \mu^{12} = 0, \quad \xi_2 \mu^{21} - \xi_1 \mu^{22} = 0. \quad (34)$$

From algebraic relations (33) and (34), we easily conclude

$$\xi_1 (\mu^{11} + \mu^{22}) = 0 \quad \text{and} \quad \xi_2 (\mu^{11} + \mu^{22}) = 0,$$

implying that the measure $\mu^{11} + \mu^{22}$ is supported on the set $\{\xi_1 = 0\} \cap \{\xi_2 = 0\} \cap P = \emptyset$, which implies $\mu^{11} + \mu^{22} \equiv 0$.

Putting $\psi \equiv 1$ in the definition of the H -distribution (formula (21)), we immediately reach to the statement of the lemma. \square

4. APPENDIX

We remind the reader of some theorems and definitions we have used in the paper.

Definition 17. For $\psi \in L^1_{loc}(\mathbb{R}^d)$ and $t > 0$ we define

$$U_t(\psi)(x) := t^{-d} \psi(xt^{-1}), \quad x \in \mathbb{R}^d.$$

If there is a bounded set $S \subset \mathbb{R}^d$, a neighborhood $N(0)$ of 0 in \mathbb{R}^d , and a $c_0 > 0$ such that

$$\int_{\mathbb{R}^d \setminus S} |U_t(\psi)(x - y) - U_t(\psi)(x)| dx \leq c_0, \quad t > 0, \quad y \in N(0),$$

then we say that ψ is a *singular kernel of exponent 1*.

Theorem 18. [26, Theorem 7.5.4] *Let ψ be a singular kernel of exponent 1. Suppose that the operator T defined on $L^2(\mathbb{R}^d)$ by $T(f) = \psi \star f$, satisfies*

$$\|T\|_{L^2 \rightarrow L^2} \leq c_0, \quad (35)$$

with constant c_0 from Definition 17. If $f \in L^1_c(\mathbb{R}^d)$, then for every $a > 0$:

$$m(\{x \in \mathbb{R}^d : |\psi \star f(x)| > a\}) \leq \frac{c_0}{a} \|f\|_{L^1(\mathbb{R}^d)},$$

where c_0 depends only on the space dimension d .

Theorem 19. [26, Theorem 5.2.9] (*Marcinkiewicz-Zygmund interpolation theorem*)
Let

$$1 \leq p_1 < p < p_2, \quad 1 \leq q_1 < q < q_2, \quad p_2 \leq q_2, \quad 0 < \alpha < 1, \\ 1/p = \alpha/p_1 + (1 - \alpha)/p_2, \quad 1/q = \alpha/q_1 + (1 - \alpha)/q_2.$$

Suppose that T is a sublinear operator mapping Lebesgue measurable functions into Lebesgue measurable functions so that there exist $M_i > 0$, $i = 1, 2$, such that for any $a > 0$ and $f \in L^{p_i}(\mathbb{R}^d)$, $i = 1, 2$:

$$m(\{x \in \mathbb{R}^d : |T(f)(x)| > a\})^{1/q_1} \leq a^{-1} M_1 \|f\|_{L^{p_1}(\mathbb{R}^d)}, \\ m(\{x \in \mathbb{R}^d : |T(f)(x)| > a\})^{1/q_2} \leq a^{-1} M_2 \|f\|_{L^{p_2}(\mathbb{R}^d)}.$$

Then, there is a finite constant M_0 depending only on p_i, q_i , $i = 1, 2$, such that

$$\|T(f)\|_{L^q(\mathbb{R}^d)} \leq M_0 M_1^\alpha M_2^{1-\alpha} \|f\|_{L^p(\mathbb{R}^d)}, \quad f \in L^p(\mathbb{R}^d).$$

Theorem 20. [26, Theorem 5.2.2] Let f be a Lebesgue measurable function defined on \mathbb{R}^d . Then:

$$m(\{x : |f(x)| > a\}) \leq a^{-p} \int_{\{x : |f(x)| > a\}} |f(x)|^p dx,$$

for every $p > 0$ and every $a > 0$.

Recall also the Calderón-Zygmund decomposition and the covering lemma:

Theorem 21. [26, Theorem 7.5.2] Let $f \in L^1(\mathbb{R}^d)$, and let $s > 0$. Then, there is a function f_0 in $L^1(\mathbb{R}^d)$, a sequence $\{f_k, k = 1, 2, \dots\}$ of functions in $L^1(\mathbb{R}^d)$ and a sequence $\{J_k, k = 1, 2, \dots\}$ of disjoint rectangles in \mathbb{R}^d such that:

- (i) $f = f_0 + \sum_{k=1}^{\infty} f_k$,
- (ii) $\|f_0\|_1 + \sum_{k=1}^{\infty} \|f_k\|_1 \leq 3\|f\|_1$,
- (iii) $|f_0(x)| \leq 2^d s$ for almost all $x \in \mathbb{R}^d$,
- (iv) $f_k(x) = 0$ for $x \in \mathbb{R}^d \setminus J_k$ and $\int_{\mathbb{R}^d} f_k(x) dx = 0$,
- (v)

$$\sum_{k=1}^{\infty} m(J_k) \leq \frac{1}{s} \int_{\mathbb{R}^d} |f(x)| dx.$$

Finally, we provide a simple lemma and its proof, which was used in the proof of Theorem 11.

Lemma 22. Let E and F be separable Banach spaces, and (b_n) an equibounded sequence of bilinear forms on $E \times F$ (more precisely, there is a constant C such that for each $n \in \mathbb{N}$ we have $|b_n(\varphi, \psi)| \leq C \|\varphi\|_E \|\psi\|_F$).

Then there exists a subsequence (b_{n_k}) and a bilinear form b (with the same bound C) such that

$$(\forall \varphi \in E)(\forall \psi \in F) \quad \lim_k b_{n_k}(\varphi, \psi) = b(\varphi, \psi).$$

Proof: To each b_n we associate a bounded linear operator $B_n : E \rightarrow F'$ by

$${}_{F'}\langle B_n \varphi, \psi \rangle_F := b_n(\varphi, \psi).$$

The above expression clearly defines a function (i.e. $B_n\varphi \in F'$ is uniquely determined), it is linear in φ , and bounded:

$$\|B_n\varphi\|_{F'} = \sup_{\psi \neq 0} \frac{|b_n(\varphi, \psi)|}{\|\psi\|_F} \leq C\|\varphi\|_E.$$

Let $\mathcal{G} \subseteq E$ be a countable dense subset; for each $\varphi \in \mathcal{G}$ the sequence $(B_n\varphi)$ is bounded in F' , so by the Banach-Alaoglu-Bourbaki theorem there is a subsequence such that

$$B_{n_1}\varphi \xrightarrow{*} \beta_1 =: B(\varphi).$$

By repeating this construction countably many times, and then applying the Cantor diagonal procedure we get a subsequence

$$(\forall \varphi \in \mathcal{G}) \quad B_{n_k}\varphi \xrightarrow{*} B(\varphi),$$

such that $\|B(\varphi)\|_{F'} \leq C\|\varphi\|_E$.

Then it is standard to extend B to a bounded linear operator on the whole space E . Clearly:

$$b(\varphi, \psi) := {}_{F'}\langle B\varphi, \psi \rangle_F = \lim_k {}_{F'}\langle B_{n_k}\varphi, \psi \rangle_F = \lim_k b_{n_k}(\varphi, \psi).$$

□

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